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Giuseppe Gentile

Department of Mathematics - University of Messina, Contrada Papardo, Salita Sperone, 31 I-98166 Messina, Italy

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Abstract

We give two sufficient conditions for a hypergroupoid to be a feeble semi-hypergroupoid and a sufficient condition to be a feeble hypergroup. We show that every hypergroup in the class of the k-quasi-Steiner hypergroupoids is feebly associative. Finally, we apply these algebraic results to study Steiner systems; in particular, we give a necessary and sufficient condition for a Steiner system to be a projective plane. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $(H,\circ)$ be a hypergroupoid, that is a non-empty set H equipped with a hyper-operation $\circ$, that is a map $\circ : H\times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ denotes the family of all non-empty subsets of H. For $A,B\subseteq H$, we define

$$A\circ B=\bigcup_{\substack{x\in A\\ y\in B}} x\circ y,$$

$$A\circ\emptyset=\emptyset\circ A=\emptyset.$$

Definition 1.1. A hypergroupoid $(H,\circ)$ is said to be a semi-hypergroup if and only if

$$\forall (x_1,x_2,x_3)\in H^3,\quad x_1\circ(x_2\circ x_3)=(x_1\circ x_2)\circ x_3.$$

A hypergroupoid $(H,\circ)$ is said to be reproductive to the right if and only if

$$\forall x\in H,\quad x\circ H=H.$$

E-mail address: gentile@dipmat.unime.it (G. Gentile).

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Analogously, we define the hypergroupoid reproductive to the left. A hypergroupoid reproductive to the right and to the left is said to be reproductive (or quasi-hypergroup). A reproductive semi-hypergroup is said to be a hypergroup.

In a hypergroupoid $(H,\circ)$ we can define a hyperproduct of n elements of H, called intersection hyperproduct, recursively in the following way (see [4]):

$$\forall x_1 \in H, \quad \overline{|\cap|}_{i=1}^{1} x_i = \{x_1\},$$
$$\forall n \geqslant 2,\ \forall (x_1,\dots,x_n) \in H^n, \quad \overline{|\cap|}_{i=1}^{n} x_i = \bigcap_{k=1}^{n-1} \left(\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \right).$$

Sometimes, we will write $\overline{|\cap|}\,(x_1,\dots,x_n)$ in place of $\overline{|\cap|}_{i=1}^{n} x_i$ to specify which elements are hypercomposed in the intersection hyperproduct. Clearly, if $(H,\circ)$ is a semi-hypergroup, the preceding definition is trivial. Now we recall the following definitions (see [4,7]):

Definition 1.2. A hypergroupoid $(H,\circ)$ is said to be a H_v-semigroup if and only if

$$\forall (x_1,x_2,x_3) \in H^3, \quad \overline{|\cap|}_{i=1}^{3} x_i \neq \emptyset,$$

that is

$$\forall (x_1,x_2,x_3) \in H^3, \quad x_1 \circ (x_2 \circ x_3) \cap (x_1 \circ x_2) \circ x_3 \neq \emptyset.$$

Definition 1.3. A hypergroupoid $(H,\circ)$ is said to be feebly associative if and only if

$$\forall n \in \mathbf{N}, \quad \forall (x_1,\dots,x_n) \in H^n, \quad \overline{|\cap|}_{i=1}^{n} x_i \neq \emptyset.$$

Clearly, any feebly associative hypergroupoid is an H_v-semigroup, but the converse, in general, is not true, as we can see in the following:

Example 1.1. Let $(H,\circ)$ be the hypergroupoid defined by the following table:

$\circ$	1	2	3	4
1	1,3	2	3,4	1,2
2	1,2	3	1,3,4	3
3	1,3	1	2,3	1,2,3
4	2,4	1,4	1,3	2,3,4

Simple calculations show that $(H,\circ)$ is a H_v-semigroup; clearly, it is reproductive to the left, but it is not reproductive to the right, since $3\circ H=\{1,2,3\}$. This hypergroupoid is not feebly associative because, if we choose $x_1=4$ and $x_2=x_3=x_4=2$, then we have

$$\overline{|\cap|}_{i=1}^{4} x_i = 4 \circ \overline{|\cap|}(2,2,2) \cap (4 \circ 2) \circ (2 \circ 2) \cap \overline{|\cap|}(4,2,2) \circ 2$$
$$= 4 \circ 1 \cap \{1,4\} \circ \{3\} \cap 1 \circ 2 = \{2,4\} \cap \{1,3,4\} \cap \{2\} = \emptyset.$$

Definition 1.4. A feebly associative hypergroupoid $(H,\circ)$ is said to be a feeble semi-hypergroup to the right if and only if

$$\forall n \in \mathbf{N}, \quad \forall (x_1,\ldots,x_n,x_{n+1}) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n} x_i,$$
$$(x \circ x_{n+1}) \cap \overline{|\cap|}_{i=1}^{n+1} x_i \neq \emptyset.$$

Analogously, $(H,\circ)$ is said to be a feeble semi-hypergroup to the left if and only if

$$\forall n \in \mathbf{N}, \quad \forall (x_0,x_1,\ldots,x_n) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n} x_i,$$
$$(x_0 \circ x) \cap \overline{|\cap|}_{i=0}^{n} x_i \neq \emptyset.$$

A feeble semi-hypergroup to the right and to the left is said to be a feeble semi-hypergroup. A reproductive feeble semi-hypergroup is said to be a feeble hypergroup.

2. On the existence of feeble semi-hypergroups and feeble hypergroups

We already know some sufficient conditions for the existence of feebly associative hypergroupoids (see [2,5]), but no condition for feeble semi-hypergroups and feeble hypergroups. In this section we investigate the existence of these hyperstructures.

Theorem 2.1. *Let $(H,\circ)$ be a hypergroupoid. If*

$$\forall x, y \in H, \quad y \in x \circ y,$$

then $(H,\circ)$ is a feeble semi-hypergroup, reproductive to the right.

Proof. Clearly, $(H,\circ)$ is reproductive to the right, because, by hypothesis, it follows that

$$\forall x \in H, \quad H = x \circ H.$$

Now we prove that $(H,\circ)$ is feebly associative; more precisely, we will show that

$$\forall n \in \mathbf{N}, \quad \forall (x_1,\ldots,x_n) \in H^n, \quad x_n \in \overline{|\cap|}_{i=1}^{n} x_i. \tag{2.1}$$

We prove (2.1) by induction on n. For $n=1$ (2.1) holds by definition of intersection hyperproduct and for $n=2$ it holds by hypothesis. Now we suppose that (2.1) holds $\forall k \leqslant n-1$ and we prove that it holds for n. We recall that, by definition of intersection hyperproduct, we have

$$\overline{|\cap|}_{i=1}^{n} x_i = \bigcap_{k=1}^{n-1} \left(\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \right).$$

By induction, it follows that

$$\forall k \in \{1,\dots,n-1\}, \quad x_n \in \overline{|\cap|}_{i=k+1}^{n} x_i,$$

and so

$$\forall k \in \{1,\dots,n-1\}, \quad \overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \supseteq \overline{|\cap|}_{i=1}^{k} x_i \circ x_n \ni x_n$$

and therefore

$$x_n \in \overline{|\cap|}_{i=1}^{n} x_i,$$

that is (2.1) holds. So, $(H,\circ)$ is feebly associative.

To prove that $(H,\circ)$ is a feeble semi-hypergroup to the right, it is enough to note that the hypothesis together with (2.1) implies

$$\forall n \in \mathbf{N}, \quad \forall (x_1,\dots,x_n,x_{n+1}) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n} x_i,$$

$$x_{n+1} \in (x \circ x_{n+1}) \cap \overline{|\cap|}_{i=1}^{n+1} x_i.$$

Now, we prove that $(H,\circ)$ is a feeble semi-hypergroup to the left. To prove this, we will show that

$$\forall n \in \mathbf{N}, \quad \forall (x_0,\dots,x_n) \in H^{n+1} \quad \Rightarrow \quad \overline{|\cap|}_{i=1}^{n} x_i \subseteq \overline{|\cap|}_{i=0}^{n} x_i. \tag{2.2}$$

We prove (2.2) by induction on n. For $n=1$ (2.2) holds by hypothesis. Now, we suppose that (2.2) holds $\forall k \leqslant n-1$ and we prove that it holds for n. Let $x \in \overline{|\cap|}_{i=1}^{n} x_i$; by definition of intersection hyperproduct, we must show that

$$\forall k \in \{0,\dots,n-1\}, \quad x \in \overline{|\cap|}_{i=0}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i. \tag{2.3}$$

By induction, we have

$$\forall k \in \{1,\dots,n-1\}, \quad \overline{|\cap|}_{i=1}^{k} x_i \subseteq \overline{|\cap|}_{i=0}^{k} x_i,$$

and so

$$\forall k \in \{1,\dots,n-1\}, \quad \overline{|\cap|}_{i=0}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \supseteq \overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i.$$

From $x \in \overline{|\cap|}_{i=1}^{n} x_i$ and by definition of intersection hyperproduct it follows that

$$\forall k \in \{1,\dots,n-1\}, \quad \overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \ni x$$

and therefore

$$\forall k \in \{1,\dots,n-1\}, \quad \overline{|\cap|}_{i=0}^{k}\, x_i \circ \overline{|\cap|}_{i=k+1}^{n}\, x_i \ni x.$$

For $k=0$, from $x \in \overline{|\cap|}_{i=1}^{n} x_i$ it follows that

$$x_0 \circ \overline{|\cap|}_{i=1}^{n}\, x_i \supseteq x_0 \circ x$$

and so

$$x_0 \circ \overline{|\cap|}_{i=1}^{n}\, x_i \ni x$$

because, by the hypothesis $x_0 \circ x \ni x$. This proves (2.2). Now from this and by hypothesis it follows that

$$\forall n \in \mathbf{N},\ \forall (x_0, x_1, \dots, x_n) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n}\, x_i,$$

$$x \in (x_0 \circ x) \cap \overline{|\cap|}_{i=0}^{n}\, x_i,$$

that is $(H,\circ)$ is a feeble semi-hypergroup to the left. The proof is now complete. □

A similar argument may be used to prove the following theorem.

Theorem 2.2. *Let $(H,\circ)$ be a hypergroupoid. If*

$$\forall x, y \in H, \quad x \in x \circ y,$$

then $(H,\circ)$ is a feeble semi-hypergroup, reproductive to the left.

Corollary 2.3. *Let $(H,\circ)$ be a hypergroupoid. If*

$$\forall x, y \in H, \quad \{x, y\} \subseteq x \circ y,$$

then $(H,\circ)$ is a feeble hypergroup.

3. Quasi-Steiner hypergroupoids

In this section we study a class of hypergroupoids, called quasi-Steiner hypergroupoids, showing that every one of these hyperstructures is feebly associative and (except for one sporadic case) not associative. First of all, we recall that a k-Steiner system is a geometric space $(H, \mathscr{L})$ where H is a non-empty set, whose elements are called points, and $\mathscr{L}$ is a family of non-empty subsets of H, whose elements are called lines, such that every line has k points and any two distinct points are contained in a unique line.

We recall the following definition and properties (see [3]):

Definition 3.1. A hypergroupoid $(H,\circ)$ is said to be a k-quasi-Steiner hypergroupoid ($k\geqslant 2$) (or simply quasi-Steiner hypergroupoid), if and only if the following conditions are satisfied:

(1) $\forall x, y \in H, \quad x \neq y, \ \{x, y\} \subseteq x \circ y$;
(2) $\forall x, y \in H, \quad |x \circ y| = k$;
(3) $\forall x, y, z, t \in H, \quad |x \circ y \cap z \circ t| > 1 \Rightarrow x \circ y = z \circ t$.

Proposition 3.1. *Any quasi-Steiner hypergroupoid $(H,\circ)$ is commutative and reproductive.*

Let $(H,\mathscr{L})$ be a k-Steiner system; we can define on H a hyperoperation by setting

$$\forall x, y \in H, \quad x \circ y = \begin{cases} l_{xy} & \text{if } x \neq y, \\ l_x & \text{if } x = y, \end{cases}$$

where l_{xy} denote the unique line troughs x and y and l_x is an arbitrary line of $\mathscr{L}$. Clearly, $(H,\circ)$ is a k-quasi-Steiner hypergroupoid that we will call associated to $(H,\mathscr{L})$.

Conversely, if $(H,\circ)$ is a k-quasi-Steiner hypergroupoid and if we consider the family

$$\mathscr{L} = \{x \circ y\}_{(x,y)\in H\times H},$$

then $(H,\mathscr{L})$ is a k-Steiner system. So, the notions of k-Steiner system and k-quasi-Steiner hypergroupoid are equivalent.

Remark 3.1. If $(H,\mathscr{L})$ is a Steiner system, with $|H|=v$, $|\mathscr{L}|=b$, then the number of quasi-Steiner hypergroupoids associated to $(H,\mathscr{L})$ is b^v.

Lemma 3.2. *Let $(H,\mathscr{L})$ be a Steiner system, $(H,\circ)$ an associated quasi-Steiner hypergroupoid; let $A\subseteq H, |A|\geqslant 2$. Then*

$$\forall x \in H, \quad x \circ A \supseteq A \subseteq A \circ x.$$

Proof. We only need to show the first inclusion, because the second one follows from the commutativity of a quasi-Steiner hypergroupoid. Two cases may occur:

1. $x \in A$;
2. $x \notin A$.

In the first case, since $|A|\geqslant 2$, then $\exists y \in A - \{x\}$, and so

$$x \circ A = x \circ x \cup \left(\bigcup_{y\in A-\{x\}} x \circ y \right).$$

By definition of a quasi-Steiner hypergroupoid, $x \neq y$ implies that

$$x \circ y \supseteq \{x, y\}$$

and therefore

$$\bigcup_{y\in A-\{x\}} x\circ y \supseteq x\cup(A-\{x\})=A.$$

In the second case, by the same definition, we have

$$x\circ A=\bigcup_{y\in A} x\circ y\supseteq A. \qquad \square$$

Lemma 3.3. *Let $(H,\mathscr{L})$ be a Steiner system, $(H,\circ)$ an associated quasi-Steiner hypergroupoid; let $x_1,x_2,x_3\in H$. If $x_r=x_s$ for some $r,s\in\{1,2,3\}, r\neq s$, then*

$$\overline{|\cap|}_{i=1}^{3} x_i\supseteq l_{x_r}.$$

Proof. We will show the inclusion in all possible cases, that is

1. $x_1=x_2=x_3$;
2. $x_1=x_2$;
3. $x_1=x_3$;
4. $x_2=x_3$.

Case 1: If we set $x_1=x_2=x_3=x$, we must show that

$$x\circ(x\circ x)\cap(x\circ x)\circ x\supseteq l_x.$$

This is true by the previous lemma.

Case 2: If we set $x_1=x_2=x$, $x_3=y$, we must show that

$$x\circ(x\circ y)\cap(x\circ x)\circ y\supseteq l_x.$$

By definition of quasi-Steiner hypergroupoid and from the previous lemma, it follows that

$$x\circ(x\circ y)=x\circ l_{xy}\supseteq x\circ x=l_x,$$
$$(x\circ x)\circ y=l_x\circ y\supseteq l_x$$

hence the claim is proved.

Case 3: If we set $x_1=x_3=x$, $x_2=y$, we must show that

$$x\circ(y\circ x)\cap(x\circ y)\circ x\supseteq l_x.$$

By definition of quasi-Steiner hypergroupoid, we have

$$x\circ(y\circ x)\supseteq x\circ x=l_x=x\circ x\subseteq(x\circ y)\circ x$$

hence the claim is proved.

Case 4: It follows from case 2 and from the commutativity of a quasi-Steiner hypergroupoid. $\square$

Lemma 3.4. *Let $(H, \mathscr{L})$ be a Steiner system, with v points; let $(H, \circ)$ be an associated quasi-Steiner hypergroupoid. Then we have*

$$\forall n \in \{1,2,\ldots,v\},\ \forall x_1,x_2,\ldots,x_n \in H, \quad \textit{pairwise distinct} \Rightarrow \{x_1,x_n\} \subseteq \overline{|\cap|}_{i=1}^{n} x_i.$$

Proof. We will show the lemma only for x_1; the proof for x_n is similar. We use induction on n.

For $n=2$ the lemma is true by definition of quasi-Steiner hypergroupoid.

Now, we suppose that the lemma is true $\forall k \leqslant n-1$ and we prove it for n, that is

$$x_1 \in \overline{|\cap|}_{i=1}^{n} x_i = \bigcap_{k=1}^{n-1} \left(\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \right).$$

By induction we have that

$$x_1 \in \overline{|\cap|}_{i=1}^{k} x_i$$

and

$$x_{k+1} \in \overline{|\cap|}_{i=k+1}^{n} x_i \quad \forall k = 1,2,\ldots,n-1$$

and therefore

$$x_1 \in x_1 \circ x_{k+1} \subseteq \overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \quad \forall k = 1,2,\ldots,n-1.$$

So the lemma is proved. $\square$

Lemma 3.5. *Let $(H, \mathscr{L})$ be a Steiner system, with v points; let $(H, \circ)$ be an associated quasi-Steiner hypergroupoid. Then we have*

$$\forall n \in \{1,2,\ldots,v\},\ \forall x_1,x_2,\ldots,x_n \in H, \quad \textit{pairwise distinct} \Rightarrow l_{x_1,x_n} \subseteq \overline{|\cap|}_{i=1}^{n} x_i.$$

Proof. We recall that, by definition of intersection hyperproduct, we have

$$\overline{|\cap|}_{i=1}^{n} x_i = \bigcap_{k=1}^{n-1} \left(\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i \right).$$

By the previous lemma it follows that

$$\forall k = 1,\ldots,n-1, \quad x_1 \in \overline{|\cap|}_{i=1}^{k} x_i, \quad x_n \in \overline{|\cap|}_{i=k+1}^{n} x_i,$$

and so

$$\forall k = 1,\ldots,n-1, \quad x_1 \circ x_n \subseteq \overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n} x_i. \qquad \square$$

Lemma 3.6. *Let $(H, \mathscr{L})$ be a Steiner system; let $(H, \circ)$ be an associated quasi-Steiner hypergroupoid and $x_1, x_2, \ldots, x_n \in H$. If $\exists r, s \in \{1, 2, \ldots, n\}$, $r \neq s$: $x_r = x_s$, then we have*

$$\overset{n}{\underset{i=1}{\overline{|\cap|}}}\, x_i \supseteq l_{x_r}.$$

Proof. We will show the lemma by induction on n. For $n = 3$, the lemma is true by Lemma 3.3. Now we suppose that the lemma is true $\forall k \leqslant n-1$ and we prove it for n.

We recall that, by definition of intersection hyperproduct, we have

$$\overset{n+1}{\underset{i=1}{\overline{|\cap|}}}\, x_i = \bigcap_{k=1}^{n} \left(\overset{k}{\underset{i=1}{\overline{|\cap|}}}\, x_i \circ \overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i \right).$$

So, it is enough to show that

$$\forall k \in \{1, \ldots, n\}, \quad \overset{k}{\underset{i=1}{\overline{|\cap|}}}\, x_i \circ \overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i \supseteq l_{x_r}. \tag{3.1}$$

We can suppose, without loss of generality, that $r < s$; then, 3 cases may occur:

1. $1 \leqslant r < s < k$,
2. $k+1 \leqslant r < s \leqslant n+1$,
3. $1 \leqslant r \leqslant k$, $k+1 \leqslant s \leqslant n+1$.

Case 1: By induction, we have

$$\overset{k}{\underset{i=1}{\overline{|\cap|}}}\, x_i \supseteq l_{x_r}.$$

Now, if $\forall i, j = k+1, \ldots, n+1$, $x_i \neq x_j$, then from Lemma 3.4 it follows that

$$x_{k+1} \in \overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i.$$

Conversely, if $\exists p, q \in \{k+1, \ldots, n+1\}$: $x_p = x_q$, then, by induction it follows that

$$\overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i \supseteq l_{x_p}.$$

So, in any case, from Lemma 3.2 it follows that (3.1) is true.

Case 2: Analogous to the previous case.

Case 3: To prove (3.1) it is enough to show that

$$x_r \in \overset{k}{\underset{i=1}{\overline{|\cap|}}}\, x_i \quad \text{or} \quad l_{x_r} \subseteq \overset{k}{\underset{i=1}{\overline{|\cap|}}}\, x_i. \tag{3.2}$$

In fact, if (3.2) holds, then we have

$$x_s \in \overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i \quad \text{or} \quad l_{x_s} \subseteq \overset{n+1}{\underset{i=k+1}{\overline{|\cap|}}}\, x_i$$

and so, exactly one of the following cases may occur

$$\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n+1} x_i \supseteq x_r \circ x_s,$$

$$\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n+1} x_i \supseteq x_r \circ l_{x_s},$$

$$\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n+1} x_i \supseteq l_{x_r} \circ x_s,$$

$$\overline{|\cap|}_{i=1}^{k} x_i \circ \overline{|\cap|}_{i=k+1}^{n+1} x_i \supseteq l_{x_r} \circ l_{x_s}.$$

Thus, (3.1) follows because of

$$x_r \circ x_s = l_{x_r}$$

and, from Lemma 3.1

$$x_r \circ l_{x_s} \supseteq l_{x_r}, \quad l_{x_r} \circ x_s \supseteq l_{x_r}, \quad l_{x_r} \circ l_{x_s} \supseteq l_{x_r}.$$

It remains to prove (3.2).

If

$$\exists t \in \{1,\dots,k\}, \quad t \neq r\colon x_t = x_r,$$

then by induction it follows that

$$l_{x_r} \subseteq \overline{|\cap|}_{i=1}^{k} x_i$$

and (3.2) is proved in this case.

Now we suppose that

$$\forall i \neq r,\ 1 \leqslant i \leqslant k, \quad x_i \neq x_r \tag{3.3}$$

and we prove that in this case

$$x_r \in \overline{|\cap|}_{i=1}^{k} x_i.$$

If $x_r \notin \overline{|\cap|}_{i=1}^{k} x_i$, by definition of intersection hyperproduct we have

$$\exists k_1 < k\colon x_r \notin \overline{|\cap|}_{i=1}^{k_1} x_i \circ \overline{|\cap|}_{i=k_1+1}^{k} x_i,$$

by Lemma 3.1, it follows that

$$x_r \notin \overline{|\cap|}_{i=1}^{k_1} x_i, \qquad x_r \notin \overline{|\cap|}_{i=k_1+1}^{k} x_i \tag{3.4}$$

or

$$\overline{|\cap|}_{i=1}^{k_1} x_i = \overline{|\cap|}_{i=k_1+1}^{k} x_i = \{x_r\}, \quad x_r \notin l_{x_r}. \tag{3.5}$$

We prove that the second case cannot occur; in fact,
if

$$\exists t, u \in \{1,\dots,k_1\} \quad \text{or} \quad \exists t, u \in \{k_1+1,\dots,k\}, \quad t \neq u\text{: } x_t = x_u$$

then, by induction, it follows that

$$\overline{|\cap|}_{i=1}^{k_1} x_i \supseteq l_{x_t}$$

and so

$$\left| \overline{|\cap|}_{i=1}^{k_1} x_i \right| \geqslant 2$$

that is, (3.5) is not possible;
if

$$\forall i, j \in \{1,\dots,k_1\},\ \forall i, j \in \{k_1,\dots,k\}, \quad i \neq j \Rightarrow x_i \neq x_j,$$

then, from Lemma 3.4, it follows that

$$x_1 \in \overline{|\cap|}_{i=1}^{k_1} x_i \quad \text{and} \quad x_{k_1} \in \overline{|\cap|}_{i=k_1+1}^{k} x_i$$

and therefore, also in this case, (3.5) is not possible because

$$\left| \overline{|\cap|}_{i=1}^{k_1} x_i \right| \geqslant 2,$$

since

$$\overline{|\cap|}_{i=1}^{k_1} x_i = \overline{|\cap|}_{i=k_1+1}^{k} x_i \supseteq \{x_1, x_{k_1+1}\} \quad \text{with } x_1 \neq x_{k_1+1}.$$

So, the only possible case in (3.4), that is
if

$$x_r \notin \overline{|\cap|}_{i=1}^{k} x_i,$$

then

$$\exists k_1 < k\text{: } x_r \notin \overline{|\cap|}_{i=1}^{k_1} x_i, \quad x_r \notin \overline{|\cap|}_{i=k_1+1}^{k} x_i.$$

Applying again this argument gives finally

$$x_r \notin x_{r-1} \circ x_r, \qquad x_r \notin x_r \circ x_{r+1}$$

and so, by definition of quasi-Steiner hypergroupoid, it follows that

$$x_r = x_{r-1}, \qquad x_r = x_{r+1}$$

which is a contradiction by 3.3. The proof is now complete. □

Now we can prove the following:

Theorem 3.7. *Let $(H, \mathscr{L})$ be a Steiner system. Then any associated quasi-Steiner hypergroupoid is feebly associative.*

Proof. We must show that

$$\forall n \in \mathbf{N}^*,\ \forall (x_1,\dots,x_n) \in H^n \ \Rightarrow\ \overline{|\cap|}_{i=1}^{n}\, x_i \neq \emptyset.$$

From Lemmas 3.5 and 3.6 it follows that
if

$$\forall i,j \in \{1,\dots,n\}, \quad i \neq j, \quad x_i \neq x_j,$$

then

$$\overline{|\cap|}_{i=1}^{n}\, x_i \supseteq l_{x_1 x_{n+1}},$$

if

$$x_r = x_s \quad \text{for some } r,s \in \{1,\dots,n\},\ r \neq s,$$

then

$$\overline{|\cap|}_{i=1}^{n}\, x_i \supseteq l_{x_r}.$$

So, in any case the theorem is proved. □

Theorem 3.8. *Let $(H, \mathscr{L})$ be a k-Steiner system with v points. If $3 \leqslant k \leqslant v-1$ then any associated quasi-Steiner hypergroupoid is not associative.*

Proof. Let $x \in H$; from $k \leqslant v-1$ it follows that there exists more than one block in $\mathscr{L}$, and therefore:

$$\exists y \neq x\colon\ y \notin l_x.$$

We will prove that

$$x \circ (x \circ y) \neq (x \circ x) \circ y.$$

The hyperproduct on the left is

$$x \circ (x \circ y) = x \circ l_{xy} = x \circ x \cup \left(\bigcup_{z \in l_{xy} - \{x\}} x \circ z \right) = l_x \cup l_{xy}.$$

So, it is enough to show that

$$\exists w \in (x \circ x) \circ y\colon w \notin l_x \cup l_{xy}. \tag{3.6}$$

We have

$$(x \circ x) \circ y = l_x \circ y = \bigcup_{t \in l_x} t \circ y. \tag{3.7}$$

If $x \in l_x$, then with $k \geqslant 3$, we have that

$$\exists t_1, t_2 \in l_x - \{x\}\colon t_1 \neq t_2;$$

if $x \notin l_x$, then

$$\forall t \in l_x \;\Rightarrow\; t \neq x.$$

So, in any case, we have

$$\exists t_1, t_2 \in l_x\colon x, t_1, t_2 \quad \text{are pairwise distinct.}$$

We have also

$$t_1 \neq y \neq t_2,$$

because $y \notin l_x$. We observe that if $t_1, t_2 \in l_{xy}$ then we have

$$l_x = l_{t_1 t_2} = l_{xy}$$

and therefore $y \in l_x$ which is a contradiction. So, we can suppose, by changing eventually the choice t_1, that is

$$t_1 \notin l_{xy}. \tag{3.8}$$

Now, from $k \geqslant 3$, it follows that

$$\exists w \in l_{t_1 y}\colon t_1 \neq w \neq y.$$

We have

$$w \in l_{t_1 y} = t_1 \circ y \quad \text{with } t_1 \in l_x$$

and therefore, by (3.7), it follows that

$$w \in (x \circ x) \circ y.$$

We will prove now that

$$w \notin l_x \cup l_{xy}.$$

1. If $w \in l_x$, then from $w \in l_{t_1 y}$ it follows that $y \in l_{t_1 w}$; so w and t_1 are two distinct points of l_x, and therefore

$$l_x = l_{t_1 w},$$

so, $y \in l_x$, which is a contradiction.

2. If $w \in l_{xy}$, then from $w \in l_{t_1 y}$ it follows that $t_1 \in l_{wy}$; so w and y are two distinct points of l_{xy}, and therefore

$$l_{xy} = l_{wy},$$

so, $t_1 \in l_{xy}$, which is a contradiction by (3.8).

So, (3.6) holds and the proof is now complete. □

We stress that the hypothesis $3 \leqslant k \leqslant v - 1$ in the preceding theorem is necessary. In fact, if $k = v$ the Steiner system $(H, \mathscr{L})$ will have only one block and so the only quasi-Steiner hypergroupoid associated to such a Steiner system is the total hypergroup defined by

$$\forall x, y \in H, \quad x \circ y = H.$$

If $k = 2$ we can consider the following counterexample:

Example 3.1. Let $(H, \mathscr{L}) \equiv S(2, 2, 4)$, that is the affine plane of order 2; then there exist exactly three hypergroups associated to $S(2, 2, 4)$. The proof is done by computer; the tables of the these hypergroups are the following:

$\circ$	1	2	3	4
1	1,2	1,2	1,3	1,4
2	1,2	1,2	2,3	2,4
3	1,3	2,3	3,4	3,4
4	1,4	2,4	3,4	3,4

$\circ$	1	2	3	4
1	1,3	1,2	1,3	1,4
2	1,2	2,4	2,3	2,4
3	1,3	2,3	1,3	3,4
4	1,4	2,4	3,4	2,4

$\circ$	1	2	3	4
1	1,4	1,2	1,3	1,4
2	1,2	2,3	2,3	2,4
3	1,3	2,3	2,3	3,4
4	1,4	2,4	3,4	1,4

4. Feeble hypergroups and projective planes

In any quasi-Steiner hypergroupoid the sufficient condition for the existence of feeble hypergroups in Corollary 2.3 is satisfied, by definition, for every pair of distinct points; so, in this case, we can rewrite Corollary 2.3 as follows:

Theorem 4.1. *Let $(H, \circ)$ be a quasi-Steiner hypergroupoid associated to a Steiner system $(H, \mathscr{L})$. If*

$$\forall x \in H, \quad x \in l_x,$$

then $(H, \circ)$ is a feeble hypergroup.

Not all quasi-Steiner hypergroupoids are feeble hypergroups; in fact, the following two theorems give sufficient conditions for the existence of quasi-Steiner hypergroupoids which are not feeble hypergroups.

Theorem 4.2. *Let $(H,\circ)$ be a quasi-Steiner hypergroupoid associated to a Steiner system $(H,\mathscr{L})$. If*

1. $\exists x \in H\colon x \in l_x$;
2. $\exists y \in l_x\colon l_x \cap l_y = \emptyset$;

then $(H,\circ)$ is not a feeble hypergroup.

Proof. We only need to show that

$$\exists n \in \mathbf{N}, \quad \exists (x_1,\dots,x_n,x_{n+1}) \in H^{n+1}, \quad \exists p \in \overline{|\cap|}_{i=1}^{n} x_i,$$

$$\overline{|\cap|}_{i=1}^{n+1} x_i \cap (p \circ x_{n+1}) = \emptyset.$$

We choose $n=2$, $x_1=x_2=x$, $x_3=y$, $p=y\in \overline{|\cap|}_{i=1}^{2} x_i = x\circ x = l_x$. Then

$$\overline{|\cap|}_{i=1}^{3} x_i = x\circ(x\circ y)\cap(x\circ x)\circ y = x\circ l_{xy}\cap l_x\circ y = (l_x\cup l_{xy})\cap(l_x\circ y),$$

but $x, y \in l_x$ with $x \neq y$ and so

$$l_x = l_{xy},$$

and therefore

$$l_x \circ y = l_x \cup l_y.$$

Finally, we have

$$\overline{|\cap|}_{i=1}^{3} x_i = l_x \cap (l_x \cup l_y) = l_x.$$

On the other hand, we have

$$p\circ x_3 = y\circ y = l_y$$

and so

$$\overline{|\cap|}_{i=1}^{3} x_i \cap (p\circ x_3) = l_x \cap l_y = \emptyset. \qquad \square$$

Theorem 4.3. *Let $(H,\circ)$ be a quasi-Steiner hypergroupoid associated to a Steiner system $(H,\mathscr{L})$. If*

1. $\exists x \in H\colon x \notin l_x$;
2. $\exists y \in l_x\colon l_y \cap l_x = \emptyset$ *and* $l_y \cap l_{xy} = \emptyset$;

then $(H,\circ)$ is not a feeble hypergroup.

Proof. As in the preceding theorem, we only need to show that

$$\exists n \in \mathbf{N}, \quad \exists (x_1, \dots, x_n, x_{n+1}) \in H^{n+1}, \quad \exists p \in \overline{|\cap|}_{i=1}^{n} x_i,$$
$$\overline{|\cap|}_{i=1}^{n+1} x_i \cap (p \circ x_{n+1}) = \emptyset.$$

We choose $n = 2$, $x_1 = x_2 = x$, $x_3 = y$, $p = y \in l_x = \overline{|\cap|}_{i=1}^{2} x_i$. Then

$$\overline{|\cap|}_{i=1}^{3} x_i = x \circ (x \circ y) \cap (x \circ x) \circ y = x \circ l_{xy} \cap l_x \circ y = (l_x \cup l_{xy}) \cap (l_x \cup l_y) = l_x,$$

and

$$p \circ x_3 = y \circ y = l_y.$$

So, it follows that

$$\overline{|\cap|}_{i=1}^{3} x_i \cap (p \circ x_3) = l_x \cap l_y = \emptyset. \qquad \square$$

Example 4.1. Let $(H, \mathscr{L}) \equiv S(2, 3, 9)$ that is the affine plane of order 3; let $(H, \circ)$ be the associated quasi-Steiner hypergroupoid defined by the following table:

$\circ$	1	2	3	4	5	6	7	8	9
1	1,5,9	1,2,3	1,2,3	1,4,7	1,5,9	1,6,8	1,4,7	1,6,8	1,5,9
2	1,2,3	3,4,8	1,2,3	2,4,9	2,5,8	2,6,7	2,6,7	2,5,8	2,4,9
3	1,2 3	1,2,3	2,4,9	3,4,8	3,5,7	3,6,9	3,5,7	3,4,8	3,6,9
4	1,4,7	2,4,9	3,4,8	1,4,7	4,5,6	4,5,6	1,4,7	3,4,8	2,4,9
5	1,5,9	2,5,8	3,5,7	4,5,6	2,6,7	4,5,6	3,5,7	2,5,8	1,5,9
6	1,6,8	2,6,7	3,6,9	4,5,6	4,5,6	1,6,8	2,6,7	1,6,8	3,6,9
7	1,4,7	2,6,7	3,5,7	1,4,7	3,5,7	2,6,7	3,5,7	7,8,9	7,8,9
8	1,6,8	2,5,8	3,4,8	3,4,8	2,5,8	1,6,8	7,8,9	3,6,9	7,8,9
9	1,5,9	2,4,9	3,6,9	2,4,9	1,5,9	3,6,9	7,8,9	7,8,9	1,2,3

This hypergroupoid satisfies the conditions of Theorem 4.2 with $x = 1$ and $y = 5$; so, it is not a feeble hypergroup.

Now we can prove the following:

Theorem 4.4. *Let* $(H, \mathscr{L})$ *be a Steiner system.* $(H, \mathscr{L})$ *is a projective plane if and only if any associated quasi-Steiner hypergroupoid is a feeble hypergroup.*

Proof. *Sufficiency*: We prove that if $(H, \mathscr{L})$ is not a projective plane, then there exists an associated quasi-Steiner hypergroupoid which is not a feeble hypergroup. This follows immediately from the preceding theorems; in fact, if $(H, \mathscr{L})$ is not a projective plane, then there exists two disjoint blocks, say l, l'; so, we can choose $l_x = l$, $l_y = l'$ and we obtain that this quasi-Steiner hypergroupoid is not a feeble hypergroup.

Necessity: From Theorem 3.7, it follows that every associated quasi-Steiner hypergroupoid $(H, \circ)$ is feebly associative and, from Proposition 3.1, that it is reproductive;

so, we only need to show that

$$\forall n \in \mathbf{N},\ \forall (x_0, x_1, \ldots, x_n) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n} x_i,$$

$$(x_0 \circ x) \cap \overline{|\cap|}_{i=0}^{n} x_i \neq \emptyset,$$

$$\forall n \in \mathbf{N},\ \forall (x_1, \ldots, x_n, x_{n+1}) \in H^{n+1}, \quad \forall x \in \overline{|\cap|}_{i=1}^{n} x_i,$$

$$(x \circ x_{n+1}) \cap \overline{|\cap|}_{i=1}^{n+1} x_i \neq \emptyset.$$

From Proposition 3.1, it follows that $(H, \circ)$ is commutative; so, it only suffices to show the second one. From Lemmas 3.5 and 3.6, it follows that

$$\forall n \in \mathbf{N},\ \forall (x_1, \ldots, x_n, x_{n+1}) \in H^{n+1}, \quad \exists l \in \mathscr{L}\colon \overline{|\cap|}_{i=1}^{n+1} x_i \supseteq l,$$

and therefore

$$\overline{|\cap|}_{i=1}^{n+1} x_i \cap (x \circ x_{n+1}) \supseteq l \cap l_{x, x_{n-1}}.$$

But in a projective plane two lines always intersect, and so

$$\overline{|\cap|}_{i=1}^{n+1} x_i \cap (x \circ x_{n+1}) \neq \emptyset.$$

The proof is now complete. □

5. For further reading

The following references are also of interest to the reader: [1,6].

References

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